1 Newton Euler Equation

We follow the definition of coordinate frames, frame transformation, and vector annotations in the main manuscript. Specifically, we use the subscribe $_{cg}$ to denote the vehicle's center of mass, subscribe $_{ee}$ to denote the tip of the end-effector. The angular velocity of the UAM is denoted as ω. Since the aerial manipulator can be treat as a rigid body, we have ${}_{\mathcal{B}}\boldsymbol{w} ={}_{\mathcal{B}} \boldsymbol{w}_{cg} ={}_{\mathcal{B}} \boldsymbol{w}_{ee}$. In addition, we define an intermediate term ${}_{\mathcal{B}}\boldsymbol{v} = [{}_{\mathcal{B}}\dot{\boldsymbol{p}}^{\top}, {}_{\mathcal{B}}\boldsymbol{\omega}^{\top}]^{\top}$ only for this derivation and is different with the $\boldsymbol{v} = [{}_{\mathcal{W}}\dot{\boldsymbol{p}}^{\top}, {}_{\mathcal{B}}\boldsymbol{\omega}^{\top}]^{\top}$ in the manuscript. $\boldsymbol{\tau}_a = [\boldsymbol{f}_a^{\top}, \boldsymbol{m}_a^{\top}]^{\top}$ and $\boldsymbol{\tau}_c = [\boldsymbol{f}_c^{\top}, \boldsymbol{m}_c^{\top}]^{\top}$ are defined as the control and contact wrench, respectively, where $\boldsymbol{f}$ and $\boldsymbol{m}$ are the corresponding force and torque components. m is the vehicle mass. $\boldsymbol{J}$ is the moment of inertia at the vehicle center of mass expressed in the body frame, i.e., $\boldsymbol{J} ={}_{\mathcal{B}} \boldsymbol{J}_{cg}$. $\boldsymbol{t} = \boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}$ is defined as the relative translation from the tip of the end-effector pointing to the vehicle center of mass in end-effector frame. We denote $[\boldsymbol{\alpha}]_{\times}$ as the skew-symmetric matrix corresponding to vector $\boldsymbol{\alpha}$. All other definitions follow the main manuscripts.

The well-known Newton-Euler equation with respect to the center of mass can be represented as:

$$\boldsymbol{M'}{}_{\mathcal{B}}\dot{\boldsymbol{v}}_{cg} + \boldsymbol{C'}{}_{\mathcal{B}}\boldsymbol{v}_{cg} = \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{f}_a \\ {}_{\mathcal{B}}\boldsymbol{m}_a \end{bmatrix} + \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{f}_c \\ {}_{\mathcal{B}}\boldsymbol{m}_c - [{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{f}_c \end{bmatrix} + \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{g} \\ \boldsymbol{0}_{3\times1} \end{bmatrix} \tag{1}$$

where

$$\boldsymbol{M'} = \begin{bmatrix} m & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{J} \end{bmatrix}, \qquad \boldsymbol{C'} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & [{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}\boldsymbol{J} \end{bmatrix} \tag{2}$$

Given the UAM structure, we have

$${}_{\mathcal{B}}\ddot{\boldsymbol{p}}_{cg} ={}_{\mathcal{B}} \ddot{\boldsymbol{p}}_{ee} + ([{}_{\mathcal{B}}\dot{\boldsymbol{\omega}}]_{\times}{}_{\mathcal{B}}\boldsymbol{t} + [{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}{}_{\mathcal{B}}\boldsymbol{t}) \tag{3}$$

We then analyze the wrench acting on the end-effector,

$$\begin{aligned} {}_{\mathcal{B}}\boldsymbol{f}_a +{}_{\mathcal{B}} \boldsymbol{f}_c +{}_{\mathcal{B}} \boldsymbol{g} &= m{}_{\mathcal{B}}\ddot{\boldsymbol{p}}_{ee} - m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\dot{\boldsymbol{\omega}} - m[{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{\omega} \\ {}_{\mathcal{B}}\boldsymbol{m}_a +{}_{\mathcal{B}} \boldsymbol{m}_c + [{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{f}_a + [{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{g} &= \boldsymbol{J}({}_{\mathcal{B}}\dot{\boldsymbol{\omega}}_{cg}) + [{}_{\mathcal{B}}\boldsymbol{\omega}_{cg}]_{\times}[\boldsymbol{J}({}_{\mathcal{B}}\boldsymbol{\omega}_{cg})] + m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times}({}_{\mathcal{B}}\ddot{\boldsymbol{p}}_{cg}) \end{aligned} \tag{4}$$

where the left-hand-side is the total wrench acting the end-effector, and the right-hand-side is the time derivative of the linear and angular momentum.

Reformulating (4), we obtain

$$\boldsymbol{M}''_{\mathcal{B}}\dot{\boldsymbol{v}}_{ee} + \boldsymbol{C}''_{\mathcal{B}}\boldsymbol{v}_{ee} = \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{f}_a \\ {}_{\mathcal{B}}\boldsymbol{m}_a + [{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{f}_a \end{bmatrix} + \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{f}_c \\ {}_{\mathcal{B}}\boldsymbol{m}_c \end{bmatrix} + \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{g} \\ [{}_{\mathcal{B}}\boldsymbol{t}]_{\times}{}_{\mathcal{B}}\boldsymbol{g} \end{bmatrix} \tag{5}$$

where

$$\boldsymbol{M''} = \begin{bmatrix} m & -m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times} \\ m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times} & \boldsymbol{J} - m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times}[{}_{\mathcal{B}}\boldsymbol{t}]_{\times} \end{bmatrix}, \qquad \boldsymbol{C''} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m[{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{B}}\boldsymbol{t}]_{\times} \\ \boldsymbol{0}_{3\times3} & [{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}\boldsymbol{J} - m[{}_{\mathcal{B}}\boldsymbol{t}]_{\times}[{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{B}}\boldsymbol{t}]_{\times} \end{bmatrix} \tag{6}$$

Notice that (5) is the well-known Newton-Euler equation with respect to a body-fixed coordinate frame whose origin does not coincident with the center of mass [1]. It is worth to mention that $\boldsymbol{M}''$ is **positive definite and symmetric**.

Next, we transform the angular velocity from the body frame to the end-effector frame. We have

$$[{}_{\mathcal{B}}\boldsymbol{\omega}]_{\times} = \boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}}[{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}, \quad [{}_{\mathcal{B}}\boldsymbol{t}]_{\times} = \boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}} \tag{7}$$

Substituting (7) into (4), we have

$$ {}_{\mathcal{E}}\boldsymbol{f}_a + {}_{\mathcal{E}}\boldsymbol{f}_c + {}_{\mathcal{E}}\boldsymbol{g} = m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}\,{}_{\mathcal{W}}\ddot{\boldsymbol{p}}_{ee} - m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\,{}_{\mathcal{E}}\dot{\boldsymbol{\omega}} - [{}_{\mathcal{E}}\boldsymbol{w}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\,{}_{\mathcal{E}}\boldsymbol{\omega} \tag{8} $$

$$ \begin{aligned} &\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\,{}_{\mathcal{B}}\boldsymbol{m}_a + \boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\,{}_{\mathcal{B}}\boldsymbol{m}_c + [{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\,{}_{\mathcal{B}}\boldsymbol{f}_a + [{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\,{}_{\mathcal{B}}\boldsymbol{g} \\ =&\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\boldsymbol{J}\boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}}\,{}_{\mathcal{E}}\dot{\boldsymbol{\omega}} + [{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\boldsymbol{J}\boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}}\,{}_{\mathcal{E}}\boldsymbol{\omega} + m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}}\,{}_{\mathcal{W}}\ddot{\boldsymbol{p}}_{ee} - m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\dot{\boldsymbol{\omega}} - m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}[{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\,{}_{\mathcal{E}}\boldsymbol{\omega} \end{aligned} \tag{9} $$

Reformulating (8) and (9), we obtain the final end-effector dynamics equation 1 in the modified manuscript:

$$ \boldsymbol{M}\dot{\boldsymbol{v}} + \boldsymbol{C}\boldsymbol{v} = \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{B}}}\,{}_{\mathcal{B}}\boldsymbol{\tau}_a + \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{C}}}\,{}_{\mathcal{C}}\boldsymbol{\tau}_c + \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{B}}}\boldsymbol{R}_{\mathcal{B}}^{\mathcal{W}}\,{}_{\mathcal{W}}\boldsymbol{g} \tag{10} $$

where

$$ \boldsymbol{v} = [{}_{\mathcal{W}}\dot{\boldsymbol{p}}^{\top}, {}_{\mathcal{B}}\boldsymbol{\omega}^{\top}]^{\top} \tag{11} $$

$$ \boldsymbol{M} = \begin{bmatrix} m\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}} & -m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times} \\ m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{W}} & \boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\boldsymbol{J}\boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}} - m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times} \end{bmatrix} \tag{12} $$

$$ \boldsymbol{C} = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m[{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times} \\ \boldsymbol{0}_{3\times3} & [{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}}\boldsymbol{J}\boldsymbol{R}_{\mathcal{B}}^{\mathcal{E}} - m[{}_{\mathcal{E}}\boldsymbol{t}]_{\times}[{}_{\mathcal{E}}\boldsymbol{\omega}]_{\times}[{}_{\mathcal{E}}\boldsymbol{t}]_{\times} \end{bmatrix} \tag{13} $$

$$ \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{B}}} = \begin{bmatrix} \boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}} & 0 \\ [{}_{\mathcal{E}}\boldsymbol{t}_{\mathcal{B}}^{\mathcal{E}}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}} & \boldsymbol{R}_{\mathcal{E}}^{\mathcal{B}} \end{bmatrix}, \quad \boldsymbol{Ad}_{\boldsymbol{T}_{\mathcal{E}}^{\mathcal{C}}} = \begin{bmatrix} \boldsymbol{R}_{\mathcal{E}}^{\mathcal{C}} & 0 \\ [{}_{\mathcal{E}}\boldsymbol{t}_{\mathcal{C}}^{\mathcal{E}}]_{\times}\boldsymbol{R}_{\mathcal{E}}^{\mathcal{C}} & \boldsymbol{R}_{\mathcal{E}}^{\mathcal{C}} \end{bmatrix} \tag{14} $$

The $\boldsymbol{M}$ matrix may look unconventional and it is not exactly symmetric. This is because our methodological choice represents velocity in the world frame and angular velocity in the end-effector frame, while all wrenches, encompassing both force and torque, are expressed in the end-effector frame.

References

[1] H. Hahn, *Rigid body dynamics of mechanisms: 1 theoretical basis*. Springer Science & Business Media, 2002.